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On De Morgan's Extension of the Algebraic Processes.

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THE algebra which I am about to consider is a symbolic algebra; that is to say, none of the processes or symbols used will have any necessary meaning. Processes will be indicated by symbols whose definitions consist in their laws of combination, and the interpretation to be given of final results will depend on what particular meaning may be given to the starting symbols. The word addition, for instance, need be nothing more than a "sound void of sense," used to indicate the process denoted by +.

The principles to which every symbolical algebra must be subjected are the following : *

1. A symbol is the representative of one process and only one.
2. Any number of processes may be looked at in their united effect as one process and represented by one symbol.
3. Every process by which we can pass from one object of contemplation to another involves a second by which we can reinstate the first object in its position; or every direct process has another which is its inverse.

The only necessary symbols are the following : =, +, —, log. All the processes of algebra can be expressed in terms of these symbols. The laws of combination to which they are subject are these :

1. $a = b$ means that wherever a occurs b may be substituted for it.
2. + and — are symbols of inverse operations; whatever one does, the other undoes; as, $a + b - b = a$. They are both subject to the following laws of (1) commutation and (2) distribution :

$$a + b - c = b - c + a = b + a - c \quad (1)$$

$$(a + b) + c = a + (b + c) \quad (2)$$

Both principles are expressed at once in the equation

$$a + (b - c) = (a - c) + b. \quad (3)$$

The word distribution is here used in a sense more general than the common one. $(a + b)c = ac + bc$ is commonly said to express *the* principle of distribution, but it is, in fact, only a particular case of it; namely, it gives the rule for distributing c over a and b when c is a multiplier and a and b are connected by the sign of addition. The index-law and the binomial theorem are rules for the distribution of c over a and b in the expressions $(ab)^c$ and $(a + b)^c$. In its most general sense, the law of distribution simply determines f, f_1, f_2 in

$$\phi [\psi(a, b), c] = f[f_1(a, c), f_2(b, c)]$$

when ϕ and ψ are given. (2) expresses the fact that if $\phi = \psi = \text{addition of}$, then we must have $f_1(a, c) = a$, $f_2(b, c) = b + c$, and $f = f_1 + f_2$. The c , in fact, disappears from either f_1 or f_2 , but so in the expansion of $(a + b)^c$, when c is negative or fractional, c is not explicitly applied to both a and b .

The expression *is a brother or sister of* obeys the above laws of commutation and distribution, and is therefore a possible meaning of $+$; but *is a father of* is not a possible meaning. The interpretation of (3) might be, "The statement that a is a brother or sister of $(b$, who is not a brother or sister of $c)$ is equivalent to the statement that $(a$, who is not a brother or sister of $c)$ is a brother or sister of b ."

3. The symbol $\log$ is defined by the equation

$$\log(a + a + \dots \text{ to } b \text{ terms}) = \log a + \log b. \quad (4)$$

It may be shown hereafter that the logarithm of a is the only function of a which possesses this property, and reason may be given for adopting the natural base; but for the present it is not necessary to attribute any meaning to the symbol $\log$ other than that contained in equation (4). This is, in fact, the way in which the idea of the logarithm was first introduced by Napier.

All the processes of algebra might be expressed in terms of these symbols alone, but for the sake of simplicity of notation other symbols, which are abbreviations of combinations of these, may be employed. We shall write for $\log(\log a)$, $\log^2 a$; for $\log \log \log a$, $\log^3 a$; etc. It will then be necessary to express the square of $\log a$ by $(\log a)^2$. By the definition already given of $+$ and $-$, we have $\log^{-n}(\log^{+n} a) = a$; or $\log^{-n}$ is a process whose effect is to annul that of $\log^n$. The expression $\log^{-n} a$ may be read *the inverse n^{th} log of a* , or *the quantity whose n^{th} log is a* . It cannot in general be taken for granted that an inverse function is a

whence, by successive substitutions,

$$a \underset{n}{+} b = \log^{-p} (\log^p a \underset{n-p}{+} \log^p b), \quad (8)$$

and in particular, when $p = n$,

$$a \underset{n}{+} b = \log^{-n} (\log^n a + \log^n b) \quad (9)$$

or

$$\log^n (a \underset{n}{+} b) = \log^n a + \log^n b, \quad (10)$$

or, in words, *the n^{th} process applied to two quantities is equal to the inverse n^{th} log of the sum of the n^{th} logs of the quantities.* (9) and (6) are the most important properties of the n^{th} process.

Taking the log of both members of (7), we have

$$\log a \underset{n-1}{+} \log b = \log (a \underset{n}{+} b).$$

If we substitute $\log^{-1} a$ for a and $\log^{-1} b$ for b , this becomes

$$\log \log^{-1} a \underset{n-1}{+} \log \log^{-1} b = \log (\log^{-1} a \underset{n}{+} \log^{-1} b),$$

or

$$a \underset{n-1}{+} b = \log (\log^{-1} a \underset{n}{+} \log^{-1} b). \quad (11)$$

So, applying (8) to $\log^{-p} a$ and $\log^{-p} b$, and taking $\log^p$ of both members of the equation, we have

$$a \underset{n-p}{+} b = \log^p (\log^{-p} a \underset{n}{+} \log^{-p} b), \quad (12)$$

and in particular, when $p = n$,

$$a + b = a \underset{0}{+} b = \log^n (\log^{-n} a \underset{n}{+} \log^{-n} b) \quad (13)$$

or

$$\log^{-n} (a \underset{0}{+} b) = \log^{-n} a \underset{n}{+} \log^{-n} b.$$

(11) gives the method of passing from any addition-process to the next lower one, and shows that there is an infinite descending series of such processes. Making $n = 0$ in (12), it becomes

$$a \underset{-p}{+} b = \log^p (\log^{-p} a + \log^{-p} b), \quad (14)$$

which gives the general expression for a negative addition process in terms of the common addition process. A negative addition process is of course a very different thing from a subtraction process. $a \underset{-1}{+} b$ bears the same relation to addition that addition bears to multiplication.

There is no essential difference between the $+$ process and the $-$ process. They obey the same laws, and either one may be taken to be the inverse of the

other. Hence the above proofs apply equally well to the $-$ process, and that also presents an infinite descending as well as ascending series. $a \underset{-n+1}{-} b$, for instance, is a command to find the $\log^{-n} b$ equal factors which when united by the $\underset{-n}{+}$ process will give a .

If we write $n + p$ for n , and therefore n for $n - p$ in (8), it becomes, using both signs,

$$a \underset{n+p}{\pm} b = \log^{-p} (\log^n a \underset{n}{\pm} \log^p b), \quad (15)$$

and this equation contains the whole theory of the positive and negative addition and subtraction processes, or better, of the positive and negative direct and inverse addition processes, provided it be observed that both n and p may be either positive or negative. In fact, (6), (11), (12), (13) are what (15) becomes when $p = 1$, $p = -1$, $p = -p$, $p = -n$, respectively.

In the $\underset{0}{+}$ process, 0 is the symbol that the same quantity has been both added and subtracted, that is, that no effect has been produced. We write $a + b - b = a + 0 = a$, and we may say that 0 is the ineffective term in the $\underset{0}{+}$ process. Making $b = \log^{-n} 0$ in (9), it becomes

$$\begin{aligned} a \underset{n}{+} \log^{-n} 0 &= \log^{-n} [\log^n a + \log^n (\log^{-n} 0)] \\ &= \log^{-n} (\log^n a + 0) = a, \end{aligned}$$

or, in the $\underset{n}{+}$ process, the ineffective term is $\log^{-n} 0$. It follows that, as by $\underset{n}{+} b$ is to be understood $0 + b$, so by $\underset{n}{+} b$ is to be understood $\log^{-n} 0 \underset{n}{+} b$.

For the sake of greater variety and sometimes greater simplicity in means of expression, the ordinary symbols of algebra may now be introduced. In accordance with the definition of the $\underset{1}{+}$ and the $\underset{2}{+}$ process, we have

$$\begin{aligned} a \underset{0}{+} b &= a + b, & a \underset{1}{+} b &= a \times b = ab, & a \underset{2}{+} b &= a^{\log b} \\ a \underset{0}{-} b &= a - b, & a \underset{1}{-} b &= a \div b = \frac{a}{b}, & a \underset{2}{-} b &= a^{\frac{1}{\log b}}. \end{aligned}$$

Introduce the symbol base by means of the equation $\log^{-1} a = e^a$. Substitute $\log^{-1} a$ for a in this equation, and it becomes

$$\log^{-1} (\log^{-1} a) = \log^{-2} a = e^{\log^{-1} a} = e^{e^a},$$

and by repeating the same process

$$\begin{aligned} \log^{-n} a &= e e \cdots e^a \quad \text{to } n \text{ exponents } e, \\ &= e e \cdots e^{\log^{-p} a} \quad \text{to } (n - p) \text{ exponents } e, \end{aligned} \quad (16)$$

The $\overset{n}{+}$ process can be expressed in one other way in terms of ordinary algebra. We have

$$\begin{aligned}
 & \frac{\log b}{b} \left[\frac{\log^2 b}{\log^3 b - 1} \left[\frac{\log^4 a - 1}{\log^3 b - 1} \right] \right] = \log^{-1} [\log b \times \log b] \\
 & = \log^{-2} \left[\frac{\log^4 a - 1}{\log^3 b - 1} - 1 \right) \log^2 b + \log^2 b] \\
 & = \log^{-2} \left(\log^2 b \frac{\log^4 a - 1}{\log^3 b - 1} \times \log^2 b = \log^{-3} [(\log^3 b - 1) \log^3 b + \log^3 b] \right) \\
 & = \log^{-4} [(\log^4 a - 1) \log^4 b + \log^4 b] = \log^{-5} (\log^5 a + \log^5 b) \\
 & = a \overset{5}{+} b.
 \end{aligned}$$

By substituting $\log a$, $\log b$ for a , b in the above expression, and taking the $\log^{-1}$ of the result, the value of $a \overset{6}{+} b$ is obtained, and hence the value of $a \overset{n}{+} b$. In the same way, it may be shown that for the inverse n^{th} process we have, if we write $\div \log b$ for $1 \div \log b$,

$$a \overset{n}{-} b = a \frac{\log a}{\left[\frac{\log^2 a}{\log^3 a - 1} \left[\frac{\div \log^{n-1} b - 1}{\log^3 a - 1} \right] \right]}$$

The $\overset{0}{+}$ process has been defined to be a process subject to the principles of commutation and distribution expressed by

$$(a + b) + c = a + (c + b). \quad (3)$$

But all the $\overset{+}{\underset{n}{+}}$ processes are subject to the same conditions, for

$$\begin{aligned}
 (a \overset{+}{\underset{n}{+}} b) \overset{+}{\underset{n}{+}} c &= \log^{-n} [\log^n (a \overset{+}{\underset{n}{+}} b) + \log^n c] \\
 &= \log^{-n} [\log^n a + \log^n b + \log^n c] \\
 &= \log^{-n} [\log^n a + (\log^n c + \log^n b)] \\
 &= \log^{-n} [\log^n a + \log^n (c \overset{+}{\underset{n}{+}} b)] \\
 &= a \overset{+}{\underset{n}{+}} (c \overset{+}{\underset{n}{+}} b)
 \end{aligned} \tag{18}$$

When $n = 1$, this becomes

$$abcde = (ab) c (de) = (abcd) e = (abe) (cd),$$

and it gives the simplest possible proof of a proposition to the demonstration of which four pages are devoted in Lejeune-Dirichlet's *Zahlentheorie*, edited by Dedekind. It depends only upon the fact that if

$$(a + b + c + d + e) = (a + b) + c + (d + e) = (a + b + e) + (c + d)$$

is true for all values of $a, b, \dots$, then it is true when for $a, b, \dots$ we substitute $\log a, \log b, \dots$.

When we reach evolution, we are in the habit of supposing that the principle of commutation no longer applies, but that is simply the fault of the unsymmetrical character of our notation. It is true that a^b is not equal to b^a , but a^b ought not to be regarded as explicitly a function of a and b , but of a and $\log^{-1} b$, and it is a function of these quantities such that a and $\log^{-1} b$ are interchangeable. It is $a^{\log(\log^{-1} b)}$ and that is equal to $(\log^{-1} b)^{\log a}$. In applying the principle to mixed direct and inverse processes, it is necessary to take account of suppressed signs. $a - b$, which is in full $+ a - b$, is, of course, not equal to $+ b - a$, but to $- b + a$. So a^b becomes, on changing the order of the terms, $(e^b)^{\log a}$.

Since it is now seen that the distinguishing mark of the $\overset{+}{\underset{0}{+}}$ process applies equally well to every process, it follows that the process of ordinary addition has not that fundamental character which we are accustomed to attribute to it. The whole theory of algebra is contained in (18), which defines any process by itself, and (5), which gives the connecting link between any process and the next higher or the next lower one. Three successive processes taken anywhere in the infinite series would serve to contain all the knowledge which has actually been expressed in terms of addition, multiplication and involution.

So far I have expressed the general process in terms of addition. It might have been expressed equally well in terms of multiplication or involution. The fact that the multiplication idea has already received greater extension than either of the other two (since $f(a)$ is properly regarded as the product of the

quantity a by the function f) might afford a reason for preferring that as the basis of the generalized idea, were it not that the existing expression for addition and its inverse, in both language and symbols, is superior to that for multiplication and its inverse. $a \pm b$ is easier to write and easier to read than $a \times b$ or $a * b$. The signs $+$ and $-$ are themselves defective. There is great advantage in having every symbol of such a form that its opposite can be indicated by actually turning it over, as Mr. Peirce's symbol for the copula, which is such that $A \prec B$ means A is wholly contained in B , and $A \succ B$ means A wholly contains B . It is an advantage possessed by many of the letters of the alphabet, but not yet made use of. For instance, a substitution and its inverse might be denoted by ψ and ϕ , and any function and its inverse by φ and ϕ . Direct and inverse operations have, in fact, the same relation to each other as the world inside and the world outside a mirror; which is to be taken as direct and which as inverse depends only on the point of view. If $+$ had had a slightly different form, say $\top$, then $-$ might have been written $\perp$, but at the time when the sign $+$ was adopted, subtraction, far from being considered as something on a par with addition, was called the *vitium negationis*, and was avoided in all possible ways. Until, then, mathematical words and signs have been made over again on scientific principles, instead of being perpetuated as they have sprung into existence by chance, $\pm$ is the best form of expression for the general process.

But it is to be observed that everything can be expressed in terms of $\times$ and $\div$ at any moment, *and without changing the index*, provided we agree to call ordinary addition the 0-addition and ordinary multiplication the 1-multiplication (and ordinary involution the 2-involution). The successive processes would be $\div$, $\times$, and $()_2^{\log()}$, and we should have *

$$\div = \times = ()_2^{\log()}.$$

It is to be observed that the inverse of $a \div b$ regarded as a function of a is not the same as its inverse regarded as a function of b . In general, the inverse

* I am not aware that any one but De Morgan has considered the extension, both forwards and backwards, of the three processes which algebra takes account of. De Morgan approaches the subject from a different point of view from that which I have here adopted. He looks at $a \div b$ and $a \times b$, and proceeds to construct new processes from the "hints" which they convey. This is doubtless the way in which new processes have usually been arrived at in the first instance, but when they have once been obtained, the most natural thing to do is to look at old and new together, abstract that part of them which is common to both, and use that as their common foundation. Formula (5), which expresses one of the two properties which each of these processes has in common with every other, has not been noticed by De Morgan at all.

De Morgan gives formulæ (6), (9), (11), (14), (16), and (18), in a notation which differs only slightly from mine. He writes indifferently $A \text{ (IV) } B$, $A \div \text{IV } B$, and $A \times \text{III } B$, for what I have called $A \div B$, and he uses the "scalar function" and its inverse, λ and χ , for what I have at once identified with the logarithm and its inverse.

of an expression regarded as a function of any quantity p which enters it (which we shall denote by I_p) is f^{-1} in $f^{-1}f(p, r) = p$, while its inverse with respect to r is f^{-1} in $f^{-1}f(p, r) = r$. For example,

$$I_a(a + b) = a - b, \quad I_b(a + b) = -a + b.$$

We have $I_a(a^b) = a^{\frac{1}{b}}$. Let us proceed to find $I_b(a^b)$; that is, a function of b such that when applied to a^b instead of b it will give b . We have

$$I_b(a^b) = I_b[(\log^{-1} b)^{\log a}].$$

This is a command to take the inverse log of b and to raise that to the power $\log a$. To invert the process (since if $f = \phi\psi$, then $f^{-1} = \psi^{-1}\phi^{-1}$ and not $\phi^{-1}\psi^{-1}$) we must raise b to the power $1 \div \log a$ and take the direct log of the result; that is,

$$I_b(a^b) = \log(b^{\frac{1}{\log a}}) = \frac{\log b}{\log a} = e^{\log^2 b - \log^2 a},$$

and in particular, when we make $a = e$, we have

$$I_b(e^b) = e^{\log^2 b}.$$

So, more generally,

$$I_a[f(a) \underset{n}{+} \phi(b)] = f^{-1}[a \underset{n}{+} \phi^{(b)}]. \quad (19)$$

If in $e^{\log^2 a - \log^2 b}$ we substitute a^b for b , we recover b , as we should.

To take another example, since

$$I_c(a^b)^c = \log c \div b \log a,$$

we have

$$\begin{aligned} I_D(e^{b \cdot D}) &= I_D f(x + h) = \log D \div h \\ &= \frac{1}{h} \left[(D - 1) - \frac{1}{2}(D - 1)^2 + \dots \right] \\ &= \frac{1}{h} \left[\log \frac{E}{e} - \frac{1}{2} \left(\log \frac{E}{e} \right)^2 + \dots \right] \end{aligned}$$

since

$$D - 1 = \log E - \log e = \log \frac{E}{e}.$$

De Morgan remarks (Trigonometry and Double Algebra, p. 166) that one operation and its inverse and the scalar function and its inverse are sufficient for expression; and he then proceeds to base the entire series of operations upon the equation of condition

$$(a \underset{n}{+} b) \underset{n+1}{+} c = (a \underset{n+1}{+} c) \underset{n}{+} (b \underset{n+1}{+} c), \quad (20)$$

which is the generalization of $(a + b)c = ac + bc$. But this is not of the nature of a definition of the $\frac{+}{n+1}$ process. It is a rule for the distribution of a term affected by the sign $\frac{+}{n+1}$ over any number of terms affected by the sign $\frac{+}{n}$, and it is of no more fundamental importance than that for the distribution of a term affected by $\frac{+}{n+2}$ over terms affected by $\frac{+}{n}$. Both are easy consequences of the definition (5). $(a + b)c = ac + bc$ does not constitute a definition of the multiplication-process; it expresses one of its properties, but not one from which all others can be deduced.

To prove (20), we have, by (5),

$$\begin{aligned}(a \frac{+}{n} b) \frac{+}{n+1} c &= (a \frac{+}{n} b) \frac{+}{n} (a \frac{+}{n} b) \frac{+}{n} \dots \text{to } \log^n c \text{ terms,} \\ &= (a \frac{+}{n} a \frac{+}{n} a \dots \text{to } \log^n c \text{ terms}) \frac{+}{n} (b \frac{+}{n} b \frac{+}{n} \dots \text{to } \log^n c \text{ terms}), \\ &= (a \frac{+}{n+1} c) \frac{+}{n} (b \frac{+}{n+1} c).\end{aligned}$$

When $n = -1, 0, +1, +2$, this gives respectively

$$\begin{aligned}\log(e^a + e^b) + c &= \log(e^{a+c} + e^{b+c}), \\ (a + b)c &= ac + bc, \\ (ab)^c &= a^c \cdot b^c, \quad \text{or} \quad c^{a+b} = c^a \cdot c^b,\end{aligned}\tag{21}$$

$$\log^3 c + \log(\log^2 a + \log^2 b) = \log(\log^2 c \log^2 a + \log^2 c \log^2 b).$$

The two equations (21), although they seem to express very different facts, do really differ from each other just as much as, and no more than, $c(a + b)$ and $(a + b)c$.

There is no method for the distribution of a term affected by any sign over terms affected by the next higher sign; for instance, $ab + c$ cannot be expressed as $\phi[f(a, c), f(b, c)]$.

De Morgan states in effect* that all the theorems of ordinary algebra which can be expressed in terms of addition and multiplication hold for all two successive processes, n and $n + 1$, if numerical coefficients are replaced by their n^{th} inverse logs. The complete proposition is this: All the theorems of ordinary algebra (which are theorems in addition, multiplication, and evolution) hold for the processes $n, n + 1, n + 2$, provided numerical summands and factors are replaced by their n^{th} inverse logs, and numerical exponents by their $(n + 1)^{\text{th}}$ inverse logs. For, given an equation expressing a theorem of ordinary algebra; it continues to be true when for the literal quantities which enter it, their n^{th} logs are substituted, and it still continues to be true when the $\log^{-n}$ of each

* Trigonometry and Double Algebra, p. 166.

member is taken. But the effect of these two operations is to substitute for $+$, $\times$, and $()^{()}$ the processes n , $n+1$, $n+2$, on the literal terms, the $\log^{-n}$ of numbers not exponents, and the $\log^{-(n+1)}$ of exponents. Say $f(b) = 5 + 4(b^3)$, then

$$\begin{aligned} f_n(b) &= \log^{-n} [5 + 4 (\log^n b)^3] \\ &= \log^{-n} [\log^n \log^{-n} 5 + \log^n \log^{-n} (4 \overline{\log^n b^3})]; \end{aligned} \quad (22)$$

but

$$\begin{aligned} \log^{-n} (4 \overline{\log^n b^3}) &= \log^{-(n+1)} [\log 4 + \log \overline{\log^n b^3}] \\ &= \log^{-(n+1)} [\log^{n+1} \log^{-n} 4 + \log^{n+1} \log^{-n} \overline{\log^n b^3}], \end{aligned} \quad (23)$$

and

$$\begin{aligned} \log^{-n} (\overline{\log^n b^3}) &= \log^{-(n+1)} [3 \log^{n+1} b] \\ &= \log^{-(n+2)} [\log 3 + \log^{n+2} b] \\ &= \log^{-(n+2)} [\log^{n+2} \log^{-(n+1)} 3 + \log^{n+2} b] \\ &= \log^{-(n+1)} 3 \underset{n+2}{+} b, \end{aligned}$$

which in (23) gives

$$\begin{aligned} \log^{-n} (4 \overline{\log^n b^3}) &= \log^{-(n+1)} [\log^{n+1} \log^{-n} 4 + \log^{n+1} (\log^{-(n+1)} 3 \underset{n+2}{+} b)] \\ &= \log^{-n} 4 \underset{n+1}{+} (\log^{-(n+1)} 3 \underset{n+2}{+} b) \end{aligned}$$

which in (22) gives

$$\begin{aligned} \log^{-n} [5 + 4 (\log^n b)^3] &= \log^{-n} \{ \log^n \log^{-n} 5 + \log^n [\log^{-n} 4 \underset{n+1}{+} (\log^{-(n+1)} 3 \underset{n+2}{+} b)] \} \\ &= \log^{-n} 5 \underset{n}{+} [\log^{-n} 4 \underset{n+1}{+} (\log^{-(n+1)} 3 \underset{n+2}{+} b)], \end{aligned} \quad (24)$$

which was to be proved. But nothing in this proof depends on the particular summand, factor, and exponent, 5, 4, and 3; hence the proposition is true in general. The reason that numbers which enter the original function as exponents are replaced by their $\log^{-n} \log^{-1}$ instead of by their $\log^{-n}$ is that the exponential function is in reality a function of the $\log^{-1}$ of the exponent, as has appeared before.

EXAMPLE. $(a + b - c)(a - b + c) = a^2 - b^2 - c^2 + 2bc$ gives in the next stage

$$\left(\frac{ab}{c} \right)^{\log(\frac{ac}{b})} = a^{\log a} \div b^{\log b} \div c^{\log c} \times 2^{\log b \log c}$$

and in general

$$(a \underset{n}{+} b \underset{n}{-} c) \underset{n+1}{+} (a \underset{n}{-} b \underset{n}{+} c) = (2 \underset{n}{+} b \underset{n+1}{+} c) \underset{n}{+} (a \underset{n+2}{+} 2 \underset{n+1}{-} b \underset{n+2}{-} c) \underset{n}{-} (c \underset{n+2}{+} 2 \underset{n+1}{-} b \underset{n+2}{-} a)$$

if we write 2 for $\log^{-n} 2$ and $\bar{2}$ for $\log^n 2$.

It is not necessary to write down the generalization of ordinary algebraical theorems. As every equation in analytical geometry is susceptible of a double interpretation according as its variables are looked at as trilinear or tangential co-ordinates, so every algebraical equation may be regarded as the receptacle of an infinite number of propositions concerning the infinite number of possible three adjacent processes.

The generalization of the binomial theorem is the law of distribution of c over a and b in $(a \underset{n}{+} b) \underset{n+2}{+} c$. We shall write $\sum_n^+$ to indicate the summation of a number of terms connected by the sign $\underset{n}{+}$; then $\sum_1^+$ denotes a product, $\sum_{\frac{1}{2}}^+$ a series of powers, etc. The binomial theorem is

$$(a + b)^p = \sum_{r=p}^0 \left[\frac{p \cdot (p-1) \dots (p-r+1)}{1 \cdot 2 \dots r} a^{p-r} b^r \right],$$

and, applying the rule for generalization, we obtain

$$(a \underset{n}{+} b) \underset{n+2}{+} c = (a \underset{n}{+} b) \underset{n+1}{+} (a \underset{n}{+} b) \underset{n+1}{+} \dots \text{to } (\log^{n+1} c, = p,) \text{ terms} \quad (25)$$

$$= \sum_{r=p}^0 \left[p \underset{n+1}{+} (p - \log^{-n} 1) \underset{n+1}{+} \dots (p - \log^{-n} r - 1) \underset{n+1}{+} \log^{-n} 1 \underset{n+1}{+} \log^{-n} 2 \dots \underset{n+1}{+} \log^{-n} r \underset{n+1}{+} (a \underset{n+2}{+} \log^{-n-1} p - r) \underset{n+1}{+} (b \underset{n+2}{+} \log^{-n-1} r) \right] \quad (26)$$

in which p is to be replaced by $\log^{n+1} c$. The immediate proof of this theorem is just as easy as that of the binomial theorem. In fact, it is exactly the same, since it consists in the application of (20) to the second number of (25). This may then be established first, and the binomial theorem taken as a special case of it.

The generalization of

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

is, if we write, with De Morgan, $\underset{n}{3}$ for $\log^{-n} 3$,

$$a \underset{n}{+} b \underset{n+2}{+} \underset{n+1}{3} = a \underset{n+2}{+} \underset{n+1}{3} \underset{n}{+} b \underset{n+2}{+} \underset{n+1}{3} \underset{n}{+} \underset{n+1}{3} \underset{n}{+} b \underset{n+1}{+} a \underset{n+2}{+} \underset{n+1}{2} \underset{n}{+} \underset{n+1}{3} \underset{n}{+} a \underset{n+1}{+} b \underset{n+2}{+} \underset{n+1}{2}.$$

For $n = 1$, this gives

$$\begin{aligned} (ab)^{\overline{\log ab^2}} &= a^{\overline{\log a^2}} \times b^{\overline{\log b^2}} \times (b^3)^{\overline{\log a^2}} \times (a^3)^{\overline{\log b^2}}, \\ &= (ab^3)^{\overline{\log a^2}} (ba^3)^{\overline{\log b^2}}, \end{aligned}$$

and, with the negative sign,

$$\left(\frac{a}{b} \right)^{\left(\log \frac{a}{b} \right)} = (a \div b^3)^{\overline{\log a^2}} \div (b \div a^3)^{\overline{\log b^2}}.$$

De Morgan says* that in the farther consideration of the higher processes new inexplicables might, and perhaps would, arise. They do in fact arise, and in the following manner. It is necessary first to establish the lemma

$$\log^p (\pm a) = \pm \log^p a. \quad (27)$$

If we make b equal to the inefficient term of the n^{th} process in

$$b \pm a = \log^{n-p} (\log^p b \pm \log^p a),$$

we have

$$\log^{-n} 0 \pm a = \pm a = \log^{-p} (\log^p \log^{-n} 0 \pm \log^p a);$$

but $\log^{p-n} 0$ is the inefficient term of the $(n-p)^{\text{th}}$ process, hence, taking $\log^p$ of both members,

$$\log^p (\pm a) = \pm \log^p a$$

and

$$\log^n (\pm a) = \pm \log^n a.$$

In particular,

$$\log (\times a) = \pm \log a, \quad \text{and} \quad \log^2 [()^a \text{ or } ()^{\frac{1}{a}}] = \pm \log^2 a.$$

We have, therefore,

$$\begin{aligned} (-a) \pm_{n+1} (-a) &= \log^{-n} [\log^n (-a) \pm_1 \log^n (-a)] \\ &= \log^{-n} [(-\log^n a) \times (-\log^n a)] \\ &= \log^{-n} [(+\log^n a) \times (+\log^n a)] \\ &= \log^{-n} [+ \log^n a \cdot \log^n a] \\ &= (+a) \pm_{n+1} (+a) = \pm (a \pm_{n+1} a) = \pm p. \end{aligned}$$

In the same way it may be shown that

$$(+a) \pm_{n+1} (-a) = - (a \pm_{n+1} a) = - p. \quad (28)$$

It appears, then, that $\pm (a \pm_{n+1} a)$ is made up of two equal terms connected by the sign $\pm_{n+1}$, being either both $(+a)$ or both $(-a)$, but $- (a \pm_{n+1} a)$ cannot possibly be made up of two equal terms connected by the sign $\pm_{n+1}$. If, then, a solution be demanded of the system of equations

$$x \pm_{n+1} y = - p, \quad x = y,$$

the answer is that the value of the variable is imaginary, and an imaginary of the n^{th} kind. In particular, the value of x in

$$x \cdot x = -p, \quad x^{\log x} = 1 \div p, \quad x^{\log x^{\log^2 x - 1}} = e^{\div \log p}$$

is an imaginary of the first, second, and third kinds respectively. But while these expressions are, in the strict sense of the term, imaginaries of a higher kind, it is none the less possible to express them as functions of $\sqrt{-1}$. The solutions of

$$x \cdot x = 0 - 1, \quad x^{\log x} = 1 \div c, \quad x^{\log x^{\log^2 x - 1}} = e^{\div e}$$

are respectively

$$x = i, \quad x = e^i, \quad x = e^{e^i}$$

and in general

$$x \underset{n+1}{+} x = \underset{n}{-} \log^{(n+1)} 0$$

gives

$$x = \log^{-n}, \quad i = ee \cdots e^i \text{ to } n \text{ e's.}$$

or the $(n+1)^{\text{th}}$ imaginary is $\log^{-n} i$.

The fact that the high imaginaries are inverse logarithms of the first imaginary corresponds to the fact that the high processes are themselves nothing more than high inverse logarithms of the sums of high logarithms.

The entire body of algebra may now be taken as proved for any process, and for any succession of processes in the infinite series. We proceed to consider differentiation. It should first be shown that, u, v , and w being functions of x ,

$$D_x(u + v + w) = D_x u + D_x v + D_x w.$$

The derivative of $\log$ should next be obtained by expanding $\log(x + \Delta x) - \log x$ in series and passing to the limit. From that the derivatives of all algebraic functions easily follow. We have

$$D_x \log x = \frac{1}{x}, \quad D_x \log^2 x = D_x \log(\log x) = \frac{D_x \log x}{\log x} = \frac{1}{x \log x},$$

$$D_x \log^n x = \frac{D_x \log^{n-1} x}{\log^{n-1} x} = \frac{D_x \log^{n-2} x}{\log^{n-2} x \cdot \log^{n-1} x} = \cdots = \frac{1}{x \cdot \log x \cdot \log^2 x \cdots \log^{n-1} x}.$$

Putting $\log^{-1} z = y$, we have $z = \log y$, $D_x z = D_x \log y = \frac{D_x y}{y}$, whence

$$D_x y = D_x \log^{-1} z = \log^{-1} x \cdot D_x z$$

and

$$D_x \log^{-1}(\log^{-1} z) = \log^{-1}(\log^{-1} z) D_x \log^{-1} z = \log^{-2} z \log^{-1} z D_x z$$

and

$$D_x \log^{-n} x = \log^{-n} x \log^{-n+1} x \cdots \log^{-1} x.$$

By this means we obtain

$$\begin{aligned} D_x(u \underset{n}{+} v) &= D_x \log^{-n} (\log^n u + \log^n v) \\ &= \log^{-n} (\log^n u + \log^n v) \cdot \log^{-n+1} (\log^n u + \log^n v) \dots \log^{-1} (\log^n u + \log^n v) \text{ into} \\ &= (u \underset{n}{+} v) (\log u \underset{n-1}{+} \log v) \dots (\log^{n-1} u \underset{1}{+} \log^{n-1} v) \text{ into} \\ &= (u \underset{n}{+} v) \log^{-1} (u \underset{n}{+} v) \log^{-2} (u \underset{n}{+} v) \dots \log^{-n+1} (u \underset{n}{+} v) \text{ into} \end{aligned}$$

where the common multiplier is

$$\left(\frac{D_x u}{u \log u \log^2 u \dots \log^{n-1} u} + \frac{D_x v}{v \log v \dots \log^{n-1} v} \right).$$

When $n = 1$,

$$D_x(u \underset{1}{+} v) = D_x(uv) = (u \underset{1}{+} v) \left(\frac{D_x u}{u} + \frac{D_x v}{v} \right) = u D_x v + v D_x u.$$

When $n = 2$,

$$\begin{aligned} D_x(u \underset{2}{+} v) &= D_x(u^{\log v}) = D_x(v^{\log u}) = u^{\log v} (\log u \cdot \log v) \left(\frac{D_x u}{u \log u} + \frac{D_x v}{v \log v} \right) \\ &= \log v \cdot u^{\log v - 1} D_x u + \log u \cdot v^{\log u - 1} D_x v. \end{aligned}$$

Substituting in this z for $\log v$, and hence $\log^{-1} z D_x z$ for $D_x v$, this becomes

$$\begin{aligned} D_x(u^z) &= z \cdot u^{z-1} D_x u + \log u \cdot \log^{-1} z^{\log u} D_x z \\ &= z \cdot u^{z-1} D_x u + \log u \cdot u^z D_x z. \end{aligned}$$

When we make $z = n$, $u = a$, $u = z = x$, $u = e$, we have respectively

$$\begin{aligned} D_x(u^n) &= n u^{n-1} D_x u, & D_x(a^z) &= \log a \cdot a^z D_x z \\ D_x(x^x) &= x^x (1 + \log x). \\ D_x(e^z) &= e^z D_x z. \end{aligned}$$

Having once obtained the value of $D_x(u + v)$ and of $D_x \log x$, we are able to avoid the necessity of ever again passing to the limit.

There is no real reason for restricting algebra to the consideration of three successive processes. The next advance should be to take in the relations of four processes, and in particular to obtain in a simple form the rule for the distribution of c in

$$\begin{aligned} (a \underset{0}{+} b) \underset{3}{+} c, &= e^{\log(a+b) \log^2 c} \\ &= c^{\log c \log^2(a+b)} \div c^{\log c}. \end{aligned}$$